Teorema 14.1(a). *Se $X \sim$ Hipergeométrica(N, n, k), a média é dada por $E(X) = \frac{nk}{N}$.*

Demonstração. Temos que

$$\begin{aligned}
E(X) &= \sum_{x=0}^{n} xP(X = x) \\
&= \sum_{x=0}^{n} x \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \\
&= 0 + \sum_{x=1}^{n} x \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=1}^{n} x \frac{\frac{k!}{x!(k-x)!}\binom{N-k}{n-x}}{\frac{N!}{n!(N-n)!}} \\
&= \sum_{x=1}^{n} x \frac{\frac{k(k-1)!}{x(x-1)!(k-x)!}\binom{N-k}{n-x}}{\frac{N(N-1)!}{n(n-1)!(N-n)!}} \\
&= \sum_{x=1}^{n} x \frac{\frac{k}{x}}{\frac{N}{n}} \frac{\frac{(k-1)!}{(x-1)!(k-x)!}\binom{N-k}{n-x}}{\frac{(N-1)!}{(n-1)!(N-n)!}} \\
&= \sum_{x=1}^{n} \cancel{x} \frac{kn}{\cancel{x}N} \frac{\binom{k-1}{x-1}\binom{N-k}{n-x}}{\binom{N-1}{n-1}} \\
&= \frac{kn}{N} \sum_{x=1}^{n} \frac{\binom{k-1}{x-1}\binom{N-k}{n-x}}{\binom{N-1}{n-1}}.
\end{aligned}$$

Fazendo $y = x - 1$, temos

$$E(X) = \frac{kn}{N} \sum_{y=0}^{n-1} \frac{\binom{k-1}{y}\binom{N-k}{n-(y+1)}}{\binom{N-1}{n-1}}.$$

Agora fazendo $M = N - 1$, $m = n - 1$ e $c = k - 1$ no somatório, temos

$$E(X) = \frac{kn}{N} \sum_{y=0}^{m} \frac{\binom{c}{y}\binom{M-c}{m-y}}{\binom{M}{m}} = \frac{kn}{N} \sum_{y=0}^{m} P(Y = y),$$

em que $Y \sim$ Hipergeométrica(M, m, c). Logo, $\sum_{y=0}^{m} P(Y = y) = 1$ e

$$E(X) = \frac{kn}{N}.$$

□

Teorema 14.1(b). *Se $X \sim$ Hipergeométrica(N, n, k), a variância é dada por*

$$Var(X) = n\frac{k}{N}\frac{(N-k)}{N}\frac{N-n}{N-1}$$

.

Demonstração. Sabemos que a variância é definida como $Var(X) = E(X^2) - [E(X)]^2$. Vamos calcular primeiro $E(X^2)$.

$$\begin{aligned}
E(X^2) &= \sum_{x=0}^{n} x^2 P(X=x) \\
&= \sum_{x=0}^{n} x^2 \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \\
&= 0 + \sum_{x=1}^{n} x^2 \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=1}^{n} x \left[x \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \right] \\
&= \sum_{x=1}^{n} x \left[\frac{kn}{N} \frac{\binom{k-1}{x-1}\binom{N-k}{n-x}}{\binom{N-1}{n-1}} \right] \\
&= \frac{kn}{N} \sum_{x=1}^{n} x \frac{\binom{k-1}{x-1}\binom{N-k}{n-x}}{\binom{N-1}{n-1}}
\end{aligned}$$

Fazendo $y = x - 1$, temos

$$E(X^2) = \frac{kn}{N} \sum_{y=0}^{n-1} (y+1) \frac{\binom{k-1}{y}\binom{N-k}{n-(y+1)}}{\binom{N-1}{n-1}}.$$

Fazendo agora $M = N-1$, $m = n-1$ e $c = k-1$ no somatório, temos

$$\begin{aligned}
E(X^2) &= \frac{kn}{N} \sum_{y=0}^{m} (y+1) \frac{\binom{c}{y}\binom{M-c}{m-y}}{\binom{M}{m}} \\
&= \frac{kn}{N} \sum_{y=0}^{m} \left[y \frac{\binom{c}{y}\binom{M-c}{m-y}}{\binom{M}{m}} + \frac{\binom{c}{y}\binom{M-c}{m-y}}{\binom{M}{m}} \right] \\
&= \frac{kn}{N} \left[\sum_{y=0}^{m} y \frac{\binom{c}{y}\binom{M-c}{m-y}}{\binom{M}{m}} + \sum_{y=0}^{m} \frac{\binom{c}{y}\binom{M-c}{m-y}}{\binom{M}{m}} \right] \\
&= \frac{kn}{N} \left[\sum_{y=0}^{m} y P(Y=y) + \sum_{y=0}^{m} P(Y=y) \right],
\end{aligned}$$

em que $Y \sim$ Hipergeométrica(M, m, c). Logo,

$$\sum_{y=0}^{m} y P(Y=y) = E(Y) = \frac{cm}{M} \quad \text{e} \quad \sum_{y=0}^{n} P(Y=y) = 1.$$

Portanto,

$$\begin{aligned}
E(X^2) &= \frac{kn}{N} \left[\frac{cm}{M} + 1 \right] \\
&= \frac{kn}{N} \left[\frac{(k-1)(n-1)}{N-1} + 1 \right]
\end{aligned}$$

Assim,

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{kn}{N}\left[\frac{(k-1)(n-1)}{N-1} + 1\right] - \left[\frac{kn}{N}\right]^2 \\
&= \frac{kn}{N}\left[\frac{kn-k-n+1}{N-1} + 1 - \frac{kn}{N}\right] \\
&= \frac{kn}{N}\left[\frac{N(kn-k-n+1) + N(N-1) - kn(N-1)}{N(N-1)}\right] \\
&= \frac{kn}{N}\left[\frac{\cancel{Nkn} - Nk - Nn + \cancel{N} + N^2 - \cancel{N} - \cancel{Nkn} + kn)}{N(N-1)}\right] \\
&= \frac{kn}{N}\left[\frac{N^2 - Nn - Nk + kn}{N(N-1)}\right] \\
&= \frac{kn}{N}\frac{(N-k)(N-n)}{N(N-1)} \\
&= n\frac{k}{N}\frac{(N-k)}{N}\frac{N-n}{N-1}
\end{aligned}
$$

□