

Teorema 15.1(a). Se $X \sim \text{Geométrica}(p)$, a média é dada por $E(X) = \frac{1}{p}$.

Demonstração. Temos que

$$\begin{aligned}
 E(X) &= \sum_{x=1}^{\infty} xP(X=x) \\
 &= \sum_{x=1}^{\infty} x(1-p)^{x-1}p \\
 &= \sum_{x=1}^{\infty} (x-1+1)(1-p)^{x-1}p \\
 &= \sum_{x=1}^{\infty} [(x-1)(1-p)^{x-1}p + (1-p)^{x-1}p] \\
 &= \sum_{x=1}^{\infty} (x-1)(1-p)^{x-1}p + \sum_{x=1}^{\infty} (1-p)^{x-1}p \\
 &= \sum_{x=1}^{\infty} (x-1)(1-p)^{x-1}p + 1
 \end{aligned}$$

Fazendo $y = x - 1$, temos

$$\begin{aligned}
 E(X) &= \sum_{y=0}^{\infty} y(1-p)^y p + 1 \\
 &= (1-p) \sum_{y=1}^{\infty} y(1-p)^{y-1} p + 1 \\
 &= (1-p)E(X) + 1
 \end{aligned}$$

Ou seja,

$$\begin{aligned}
 E(X) &= (1-p)E(X) + 1 \\
 E(X) - (1-p)E(X) &= 1 \\
 \cancel{E(X)} - \cancel{E(X)} + pE(X) &= 1 \\
 E(X) &= \frac{1}{p}
 \end{aligned}$$

□

Teorema 15.1(b). Se $X \sim \text{Geométrica}(p)$, a variância é dada por $\text{Var}(X) = \frac{1-p}{p^2}$.

Demonstração. Sabemos que a variância é definida como $\text{Var}(X) = E(X^2) - [E(X)]^2$.

Vamos calcular primeiro $E(X^2)$.

$$\begin{aligned}
 E(X^2) &= \sum_{x=1}^{\infty} x^2 P(X=x) \\
 &= \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} p \\
 &= \sum_{x=1}^{\infty} (x-1+1)^2 (1-p)^{x-1} p \\
 &= \sum_{x=1}^{\infty} [(x-1)^2 + 2(x-1) + 1] (1-p)^{x-1} p \\
 &= \sum_{x=1}^{\infty} (x-1)^2 (1-p)^{x-1} p + \sum_{x=1}^{\infty} 2(x-1) (1-p)^{x-1} p + \sum_{x=1}^{\infty} (1-p)^{x-1} p
 \end{aligned}$$

Fazendo $y = x - 1$ e como $\sum_{x=1}^{\infty} (1-p)^{x-1} p = 1$, temos

$$\begin{aligned}
 E(X^2) &= \sum_{y=0}^{\infty} y^2 (1-p)^y p + \sum_{y=0}^{\infty} 2y (1-p)^y p + 1 \\
 &= 0 + \sum_{y=1}^{\infty} y^2 (1-p)^y p + 0 + \sum_{y=1}^{\infty} 2y (1-p)^y p + 1 \\
 &= (1-p) \sum_{y=1}^{\infty} y^2 (1-p)^{y-1} p + 2(1-p) \sum_{y=1}^{\infty} y (1-p)^{y-1} p + 1 \\
 &= (1-p) E(X^2) + 2(1-p) E(X) + 1
 \end{aligned}$$

Usando $E(X) = \frac{1}{p}$, temos

$$\begin{aligned}
 E(X^2) &= (1-p) E(X^2) + 2(1-p) \frac{1}{p} + 1 \\
 E(X^2) - (1-p) E(X^2) &= \frac{2(1-p) + p}{p} \\
 \cancel{E(X^2)} - \cancel{E(X^2)} + p E(X^2) &= \frac{2 - 2p + p}{p} \\
 E(X^2) &= \frac{2-p}{p^2}
 \end{aligned}$$

o que dá o resultado

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2-p}{p^2} - \left[\frac{1}{p} \right]^2 = \frac{2-p-1}{p^2} = \frac{1-p}{p^2}$$

□

Teorema 15.2(a). Se $X \sim \text{Binomial Negativa}(r, p)$, a média é dada por $E(X) = \frac{r}{p}$.

Demonstração. Temos que

$$\begin{aligned}
E(X) &= \sum_{x=r}^{\infty} xP(X=x) \\
&= \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x \frac{(x-1)!}{(x-r)!(r-1)!} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \frac{x!}{(x-r)!(r-1)!} \left[\frac{r}{r} \right] p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \frac{x!}{(x-r)! r!} r p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \binom{x}{r} r \left[\frac{p}{p} \right] p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \binom{x}{r} \frac{r}{p} p^{r+1} (1-p)^{x-r} \\
&= \frac{r}{p} \sum_{x=r}^{\infty} \binom{x}{r} p^{r+1} (1-p)^{x-r}
\end{aligned}$$

Fazendo $x = y - 1$, temos

$$E(X) = \frac{r}{p} \sum_{y=r+1}^{\infty} \binom{y-1}{r} p^{r+1} (1-p)^{(y-1)-r}$$

Agora fazendo $r = r + 1$, temos

$$\begin{aligned}
E(X) &= \frac{r}{p} \sum_{y=s}^{\infty} \binom{y-1}{s-1} p^s (1-p)^{(y-1)-(s-1)} \\
&= \frac{r}{p} \sum_{y=s}^{\infty} \binom{y-1}{s-1} p^s (1-p)^{y-s} \\
&= \frac{r}{p} \sum_{y=s}^{\infty} P(Y=y), \text{ com } Y \sim \text{Binomial Negativa}(s, p) \\
&= \frac{r}{p} \cdot 1 \\
&= \frac{r}{p}
\end{aligned}$$

□

Teorema 15.2(b). Se $X \sim \text{Binomial Negativa}(r, p)$, a variância é dada por $\text{Var}(X) = \frac{r(1-p)}{p^2}$.

Demonstração. Sabemos que a variância é definida como $\text{Var}(X) = E(X^2) - [E(X)]^2$. Vamos calcular primeiro $E(X^2)$.

$$\begin{aligned}
E(X^2) &= \sum_{x=1}^{\infty} x^2 P(X=x) \\
&= \sum_{x=r}^{\infty} x^2 \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x^2 \frac{(x-1)!}{(x-r)!(r-1)!} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x^2 \frac{(x-1)!}{(x-r)!(r-1)!} \left[\frac{rp}{rp} \right] p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x \frac{x(x-1)!}{(x-r)!r(r-1)!} \left[\frac{r}{p} \right] p p^r (1-p)^{x-r} \\
&= \frac{r}{p} \sum_{x=r}^{\infty} x \frac{x!}{(x-r)!r!} p^{r+1} (1-p)^{x-r} \\
&= \frac{r}{p} \sum_{x=r}^{\infty} x \binom{x}{r} p^{r+1} (1-p)^{x-r}
\end{aligned}$$

Fazendo $x = y - 1$, temos

$$E(X^2) = \frac{r}{p} \sum_{y=r+1}^{\infty} (y-1) \binom{y-1}{r} p^{r+1} (1-p)^{(y-1)-r}$$

Fazendo agora $s = r + 1$, obtemos

$$\begin{aligned}
E(X^2) &= \frac{r}{p} \sum_{y=s}^{\infty} (y-1) \binom{y-1}{s-1} p^s (1-p)^{(y-1)-(s-1)} \\
&= \frac{r}{p} \sum_{y=s}^{\infty} (y-1) \binom{y-1}{s-1} p^s (1-p)^{y-s} \\
&= \frac{r}{p} \left[\sum_{y=s}^{\infty} y \binom{y-1}{s-1} p^s (1-p)^{y-s} - \sum_{y=s}^{\infty} \binom{y-1}{s-1} p^s (1-p)^{y-s} \right] \\
&= \frac{r}{p} \left[E(Y) - \sum_{y=s}^{\infty} P(Y=y) \right],
\end{aligned}$$

em que $Y \sim \text{Binomial Negativa}(s, p)$. Logo, $E(Y) = \frac{s}{p} = \frac{r+1}{p}$ e $\sum_{y=s}^{\infty} P(Y=y) = 1$. Daí,

$$E(X^2) = \frac{r}{p} \left[\frac{r+1}{p} - 1 \right] = \frac{r}{p} \left[\frac{r+1-p}{p} \right] = \frac{r^2 + r - rp}{p^2}$$

Portanto,

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{r^2 + r - rp}{p^2} - \left[\frac{r}{p} \right]^2 \\ &= \frac{\cancel{r^2} + r - rp - \cancel{r^2}}{p^2} \\ &= \frac{r(1-p)}{p^2} \end{aligned}$$

□