Teorema 16.1(a). *Se $X \sim Poisson(\lambda)$, a média é dada por $E(X) = \lambda$.*

Demonstração. Temos que

$$\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} xP(X = x) \\
&= \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} \\
&= 0 + \sum_{x=1}^{\infty} \frac{xe^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} \frac{\cancel{x}e^{-\lambda}\lambda^x}{\cancel{x}(x-1)!} \\
&= e^{-\lambda}\sum_{x=1}^{\infty} \frac{\lambda\lambda^{x-1}}{(x-1)!} \\
&= \lambda e^{-\lambda}\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}
\end{aligned}$$

Fazendo $y = x - 1$, temos

$$E(X) = \lambda e^{-\lambda}\sum_{y=1}^{\infty} \frac{\lambda^y}{y!}$$

Como $\sum_{y=1}^{\infty} \frac{\lambda^y}{y!} = e^{\lambda}$,

$$E(X) = \lambda \cancel{e^{-\lambda}}\cancel{e^{\lambda}} = \lambda$$

□

Teorema 16.1(b). *Se $X \sim Poisson(\lambda)$, a variância é dada por $Var(X) = \lambda$.*

Demonstração. Sabemos que a variância é definida como $Var(X) = E(X^2) - [E(X)]^2$. Vamos calcular primeiro $E(X^2)$.

$$\begin{aligned}
E(X^2) &= \sum_{x=0}^{\infty} x^2P(X = x) \\
&= \sum_{x=0}^{\infty} x^2\frac{e^{-\lambda}\lambda^x}{x!} \\
&= 0 + \sum_{x=1}^{\infty} x^2\frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} x^{\cancel{2}}\frac{e^{-\lambda}\lambda^x}{\cancel{x}(x-1)!} \\
&= \sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda\lambda^{x-1}}{(x-1)!} \\
&= \lambda\sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}
\end{aligned}$$

Fazendo $y = x - 1$, temos

$$\begin{aligned} E(X^2) &= \lambda \sum_{y=0}^{\infty} (y+1) \frac{e^{-\lambda}\lambda^y}{y!} \\ &= \lambda \left[\sum_{y=0}^{\infty} y \frac{e^{-\lambda}\lambda^y}{y!} + \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \right] \\ &= \lambda \left[E(X) + \sum_{y=0}^{\infty} P(Y=y) \right] \end{aligned}$$

em que $Y \sim \text{Poisson}(\lambda)$, logo

$$\begin{aligned} E(X^2) &= \lambda \left[E(X) + 1 \right] \\ &= \lambda \left[\lambda + 1 \right]. \end{aligned}$$

Daí,

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= \lambda \left[\lambda + 1 \right] - \lambda^2 \\ &= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} \\ &= \lambda \end{aligned}$$

$\square$

Teorema 16.2(a). *Se $X \sim$ Binomial Negativa(r,p), a média é dada por $E(X) = \frac{r}{p}$.*

Demonstração. Temos que

$$\begin{aligned}
E(X) &= \sum_{x=r}^{\infty} xP(X=x) \\
&= \sum_{x=r}^{\infty} x\binom{x-1}{r-1}p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x\frac{(x-1)!}{(x-r)!(r-1)!}p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \frac{x!}{(x-r)!(r-1)!}\left[\frac{r}{r}\right]p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \frac{x!}{(x-r)!\,r!}\,r\,p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \binom{x}{r} r\left[\frac{p}{p}\right]p^r(1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} \binom{x}{r}\frac{r}{p}\,p^{r+1}(1-p)^{x-r} \\
&= \frac{r}{p}\sum_{x=r}^{\infty} \binom{x}{r}p^{r+1}(1-p)^{x-r}
\end{aligned}$$

Fazendo $x = y - 1$, temos

$$E(X) = \frac{r}{p}\sum_{y=r+1}^{\infty}\binom{y-1}{r}p^{r+1}(1-p)^{(y-1)-r}$$

Agora fazendo $r = r + 1$, temos

$$\begin{aligned}
E(X) &= \frac{r}{p}\sum_{y=s}^{\infty}\binom{y-1}{s-1}p^{s}(1-p)^{(y-1)-(s-1)} \\
&= \frac{r}{p}\sum_{y=s}^{\infty}\binom{y-1}{s-1}p^{s}(1-p)^{y-s} \\
&= \frac{r}{p}\sum_{y=s}^{\infty}P(Y=y),\ \text{com } Y \sim \text{Binomial Negativa}(s,p) \\
&= \frac{r}{p}\cdot 1 \\
&= \frac{r}{p}
\end{aligned}$$

$\square$

Teorema 16.2(b). *Se $X \sim$ Binomial Negativa(r, p), a variância é dada por* $Var(X) = \frac{r(1-p)}{p^2}$.

Demonstração. Sabemos que a variância é definida como $Var(X) = E(X^2) - [E(X)]^2$. Vamos calcular primeiro $E(X^2)$.

$$
\begin{aligned}
E(X^2) &= \sum_{x=1}^{\infty} x^2 P(X = x) \\
&= \sum_{x=r}^{\infty} x^2 \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x^2 \frac{(x-1)!}{(x-r)!(r-1)!} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x^2 \frac{(x-1)!}{(x-r)!(r-1)!} \left[\frac{rp}{rp}\right] p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} x \frac{x(x-1)!}{(x-r)!\, r(r-1)!} \left[\frac{r}{p}\right] p p^r (1-p)^{x-r} \\
&= \frac{r}{p} \sum_{x=r}^{\infty} x \frac{x!}{(x-r)!\, r!} p^{r+1} (1-p)^{x-r} \\
&= \frac{r}{p} \sum_{x=r}^{\infty} x \binom{x}{r} p^{r+1} (1-p)^{x-r}
\end{aligned}
$$

Fazendo $x = y - 1$, temos

$$
E(X^2) = \frac{r}{p} \sum_{y=r+1}^{\infty} (y-1) \binom{y-1}{r} p^{r+1} (1-p)^{(y-1)-r}
$$

Fazendo agora $s = r + 1$, obtemos

$$
\begin{aligned}
E(X^2) &= \frac{r}{p} \sum_{y=s}^{\infty} (y-1) \binom{y-1}{s-1} p^s (1-p)^{(y-1)-(s-1)} \\
&= \frac{r}{p} \sum_{y=s}^{\infty} (y-1) \binom{y-1}{s-1} p^s (1-p)^{y-s} \\
&= \frac{r}{p} \left[\sum_{y=s}^{\infty} y \binom{y-1}{s-1} p^s (1-p)^{y-s} - \sum_{y=s}^{\infty} \binom{y-1}{s-1} p^s (1-p)^{y-s}\right] \\
&= \frac{r}{p} \left[E(Y) - \sum_{y=s}^{\infty} P(Y = y)\right],
\end{aligned}
$$

em que $Y \sim$ Binomial Negativa(s, p). Logo, $E(Y) = \frac{s}{p} = \frac{r+1}{p}$ e $\sum\limits_{y=s}^{\infty} P(Y = y) = 1$. Daí,

$$
E(X^2) = \frac{r}{p} \left[\frac{r+1}{p} - 1\right] = \frac{r}{p} \left[\frac{r+1-p}{p}\right] = \frac{r^2 + r - rp}{p^2}
$$

Portanto,

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{r^2 + r - rp}{p^2} - \left[\frac{r}{p}\right]^2 \\
&= \frac{\cancel{r^2} + r - rp - \cancel{r^2}}{p^2} \\
&= \frac{r(1-p)}{p^2}
\end{aligned}
$$

□